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BEE604 – Digital Signal Processing

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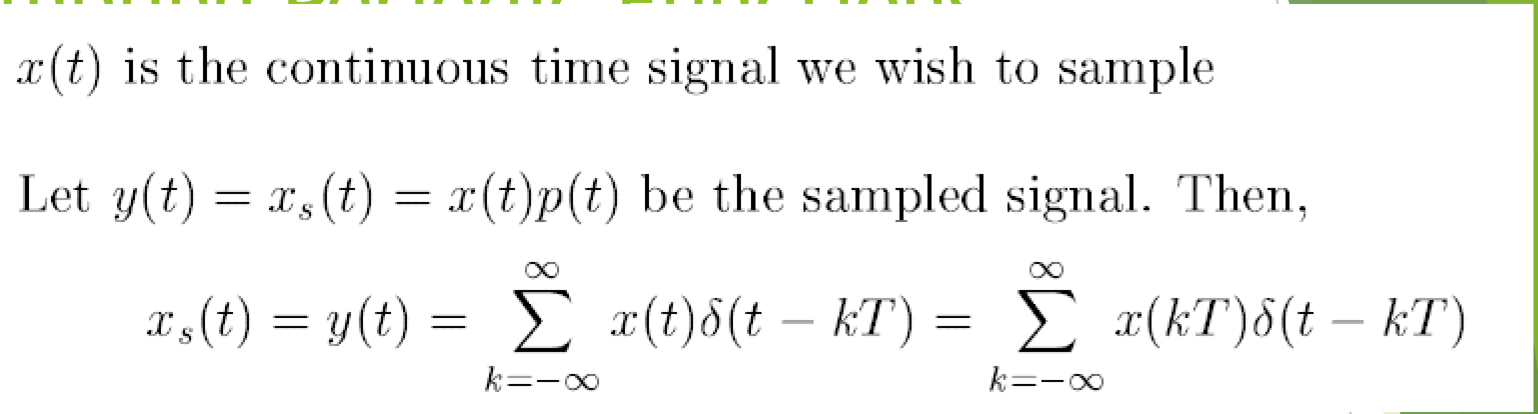
CONTENTS

- Sampling
- Discrete Time Fourier Transform
- Properties of DTFT
- Discrete Fourier Transform
- Inverse Discrete Fourier Transform
- FAST FOURIER TRANSFORMS

Sampling

- Continuous signals are digitized using digital computers
- When we sample, we calculate the value of the continuous signal at discrete points
  - How fast do we sample
  - What is the value of each point
- Quantization determines the value of each samples value

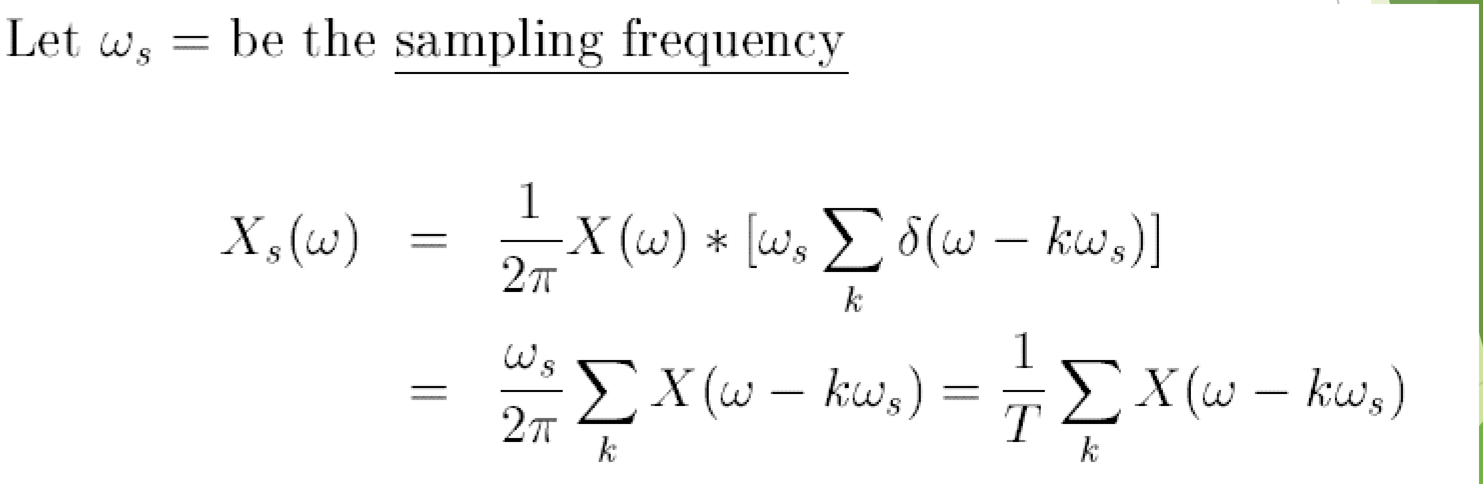
Sampling Periodic Functions



$x(t)$ is the continuous time signal we wish to sample

Let $y(t) = x_s(t) = x(t)p(t)$ be the sampled signal. Then,

$$x_s(t) = y(t) = \sum_{k=-\infty}^{\infty} x(t)\delta(t-kT) = \sum_{k=-\infty}^{\infty} x(kT)\delta(t-kT)$$



Let $\omega_s =$ be the <u>sampling frequency</u>

$$\begin{aligned} X_s(\omega) &= \frac{1}{2\pi} X(\omega) * [\omega_s \sum_k \delta(\omega - k\omega_s)] \\ &= \frac{\omega_s}{2\pi} \sum_k X(\omega - k\omega_s) = \frac{1}{T} \sum_k X(\omega - k\omega_s) \end{aligned}$$

- Note that wb = Bandwidth, thus if $\omega_s - \omega_b < \omega_b$ then aliasing occurs (signal overlaps)

-To avoid aliasing $\omega_s - \omega_b > \omega_b \text{ or } \omega_s > 2\omega_b$

-According sampling theory: $\omega_s > 2\omega_b$

To hear music up to 20KHz a CD should sample at the rate of 44.1 KHz

Discrete Time Fourier Transform

- In likely we only have access to finite amount of data sequences (after

$$x_s(t) \leftrightarrow \sum_{n=-\infty}^{\infty} x(nT)e^{-jn\omega T}$$

- Recall for continu hen the signal is sam

$$x(nT) = x[n] \qquad \Omega = \omega T$$

- Assuming

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

- Discrete-Time Fourier Transform (DTFT):

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega$$

Discrete Time Fourier Transform

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \qquad x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega$$

- Discrete-Time Fourier Transform (DTFT):

$$e^{j\Omega} = e^{j(\Omega+2\pi)} = e^{j\Omega}e^{j2\pi} = e^{j\Omega}$$

- A few points
  - DTFT is periodic in frequency with period of 2π
  - X[n] is a discrete signal
  - DTFT allows us to find the spectrum of the discrete signal as viewed from a window

Example of Convolution

- (

$$x[n] = \sum_{k=-\infty}^{\infty} x_0[n-kN] = \sum_{k=-\infty}^{\infty} x_0[n] * \delta[n-kN] = x_0[n] * \sum_{k=-\infty}^{\infty} \delta[n-kN]$$

  - We can write x[n] (a periodic function) as an infinite sum of the function $x_0[n]$ (a non-periodic function) shifted N units at a time

$$x[n] = x_0[n] * p[n] \longleftrightarrow X_0(\Omega)P(\Omega) = X(\Omega)$$

  - This wi

$$p[n] = \sum_{k=-\infty}^{\infty} \delta[n-kN] \leftrightarrow \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta(\Omega - \frac{2\pi k}{N}) = P(\Omega)$$

$$\begin{aligned} X(\Omega) &= X_0(\Omega)\left(\frac{2\pi}{N}\sum_k \delta(\Omega - \frac{2\pi k}{N})\right) \\ &= \frac{2\pi}{N}\sum_k X_0(\frac{2\pi k}{N})\delta(\Omega - \frac{2\pi k}{N}) \end{aligned}$$

  - Thus

$$\boxed{x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_0(\frac{2\pi k}{N})e^{\frac{j2\pi kn}{N}}}$$

Finding DTFT For periodic signals

$$x_0[n] = \begin{cases} x[n], & 0 \le n \le N-1 \\ 0, & \text{otherwise.} \end{cases}$$

- Starting with xo[n]

$$\boxed{X_0(\Omega) = \sum_{n=-\infty}^{\infty} x_0[n]e^{-jn\Omega} = \sum_{n=0}^{N-1} x_0[n]e^{-jn\Omega}}$$

- DTFT of xo[n]

$$X(\Omega) = X_0(\Omega)\left(\frac{2\pi}{N}\sum_k \delta(\Omega - \frac{2\pi k}{N})\right)$$

$$\boxed{= \frac{2\pi}{N}\sum_k X_0(\frac{2\pi k}{N})\delta(\Omega - \frac{2\pi k}{N})}$$

DT Fourier Transform

1. Ω is in radian and it is between 0 and 2π in each discrete time interval
2. This is different from ω where it was between – INF and + INF
3. Note that $X(\Omega)$ is periodic

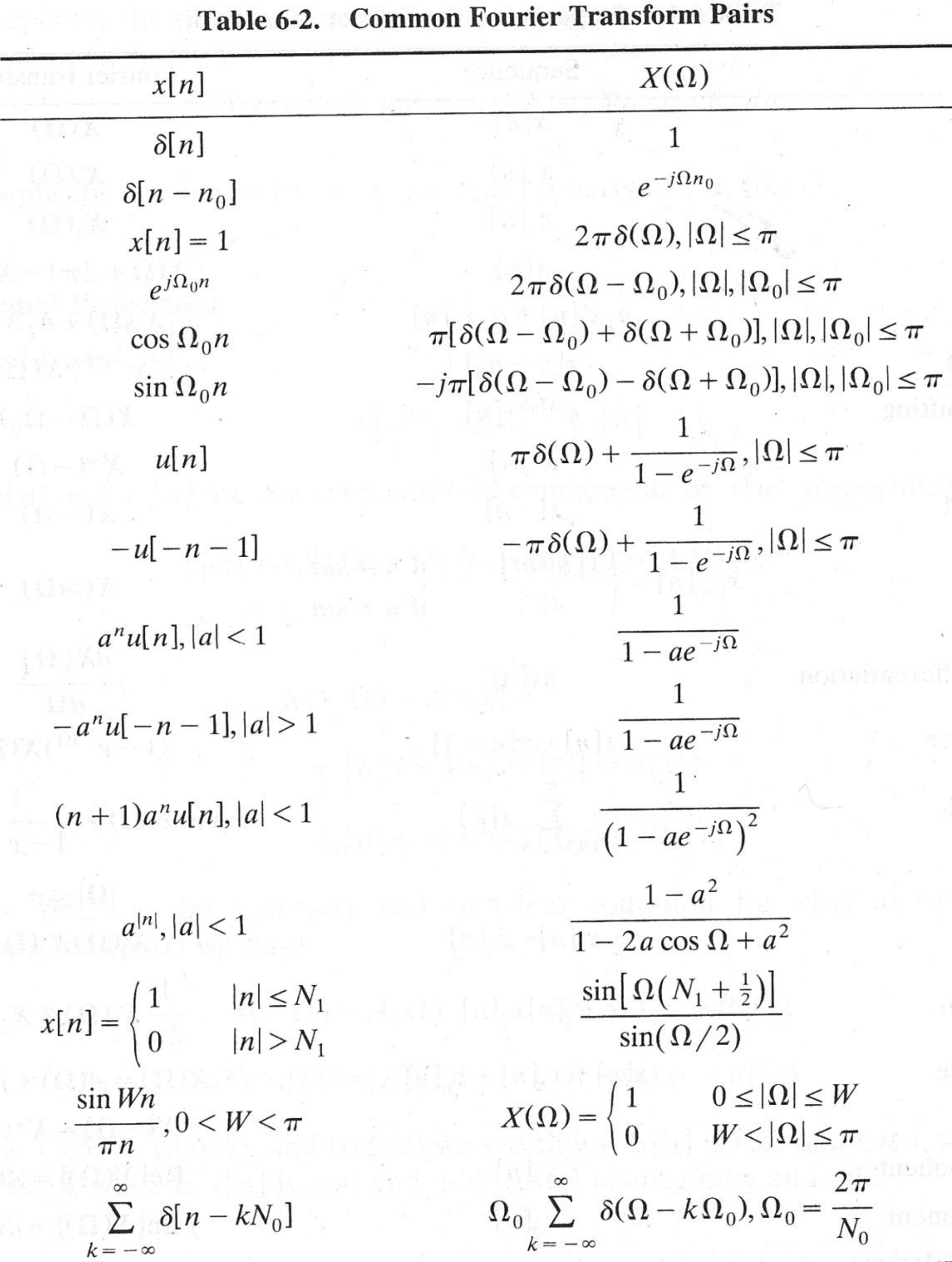


Table 6-2. Common Fourier Transform Pairs

$x[n]$	$X(\Omega)$
$\delta[n]$	1
$\delta[n-n_0]$	$e^{-j\Omega n_0}$
$x[n]=1$	$2\pi\delta(\Omega), \lvert\Omega\rvert \le \pi$
$e^{j\Omega_0 n}$	$2\pi\delta(\Omega-\Omega_0), \lvert\Omega\rvert, \lvert\Omega_0\rvert \le \pi$
$\cos \Omega_0 n$	$\pi[\delta(\Omega-\Omega_0)+\delta(\Omega+\Omega_0)], \lvert\Omega\rvert, \lvert\Omega_0\rvert \le \pi$
$\sin \Omega_0 n$	$-j\pi[\delta(\Omega-\Omega_0)-\delta(\Omega+\Omega_0)], \lvert\Omega\rvert, \lvert\Omega_0\rvert \le \pi$
$u[n]$	$\pi\delta(\Omega)+\dfrac{1}{1-e^{-j\Omega}}, \lvert\Omega\rvert \le \pi$
$-u[-n-1]$	$-\pi\delta(\Omega)+\dfrac{1}{1-e^{-j\Omega}}, \lvert\Omega\rvert \le \pi$
$a^n u[n], \lvert a\rvert < 1$	$\dfrac{1}{1-ae^{-j\Omega}}$
$-a^n u[-n-1], \lvert a\rvert > 1$	$\dfrac{1}{1-ae^{-j\Omega}}$
$(n+1)a^n u[n], \lvert a\rvert < 1$	$\dfrac{1}{(1-ae^{-j\Omega})^2}$
$a^{\lvert n\rvert}, \lvert a\rvert < 1$	$\dfrac{1-a^2}{1-2a\cos\Omega+a^2}$
$x[n]=\begin{cases}1 & \lvert n\rvert \le N_1 \\ 0 & \lvert n\rvert > N_1\end{cases}$	$\dfrac{\sin\left[\Omega\left(N_1+\frac{1}{2}\right)\right]}{\sin(\Omega/2)}$
$\dfrac{\sin Wn}{\pi n}, 0 < W < \pi$	$X(\Omega)=\begin{cases}1 & 0 \le \lvert\Omega\rvert \le W \\ 0 & W < \lvert\Omega\rvert \le \pi\end{cases}$
$\sum_{k=-\infty}^{\infty} \delta[n-kN_0]$	$\Omega_0 \sum_{k=-\infty}^{\infty} \delta(\Omega-k\Omega_0), \Omega_0=\dfrac{2\pi}{N_0}$

Properties of DTF

- Remember:
- For time scaling note that m>1 → Signal spreading

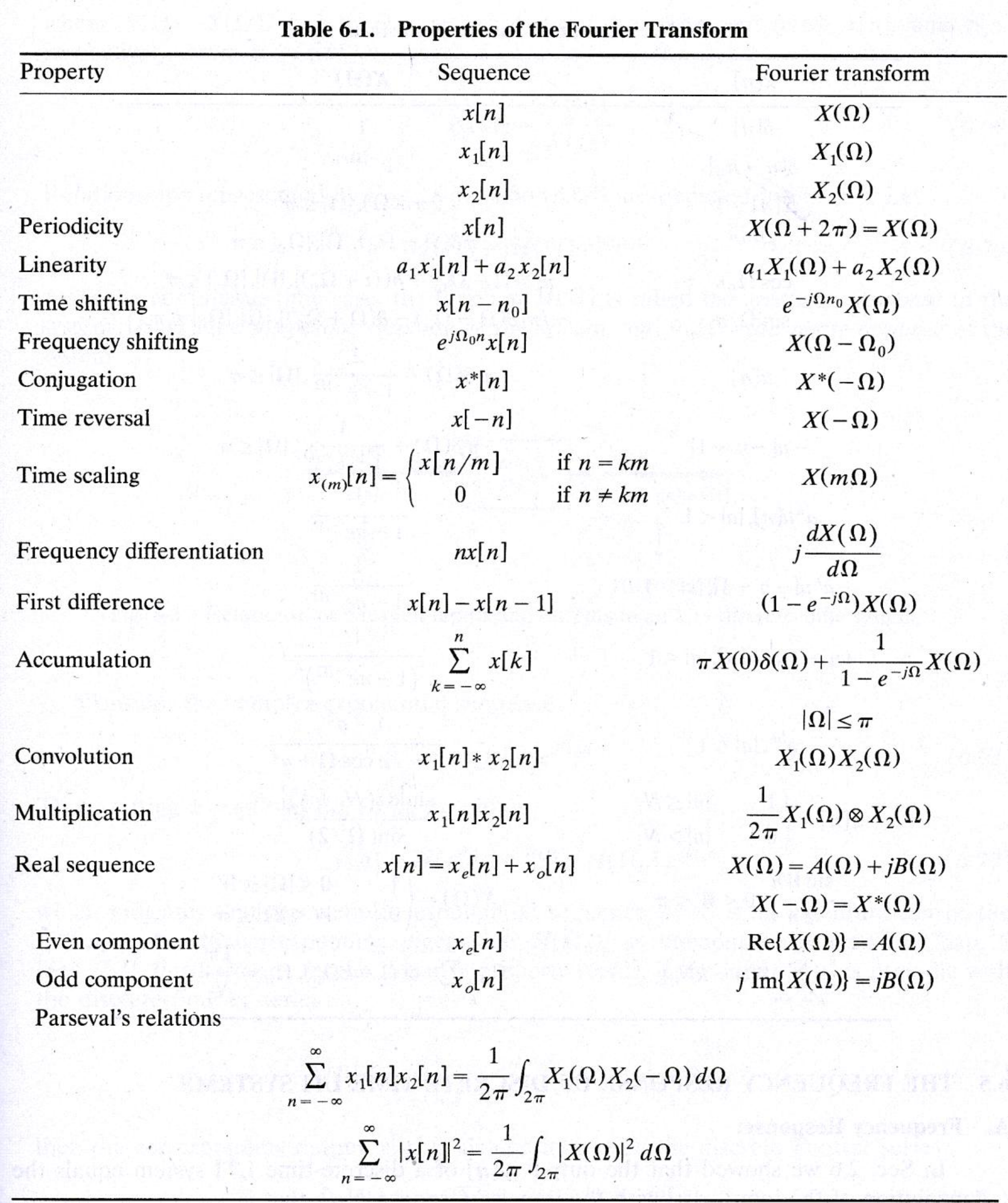


Table 6-1. Properties of the Fourier Transform

Property	Sequence	Fourier transform
	$x[n]$	$X(\Omega)$
	$x_1[n]$	$X_1(\Omega)$
	$x_2[n]$	$X_2(\Omega)$
Periodicity	$x[n]$	$X(\Omega+2\pi)=X(\Omega)$
Linearity	$a_1x_1[n]+a_2x_2[n]$	$a_1X_1(\Omega)+a_2X_2(\Omega)$
Time shifting	$x[n-n_0]$	$e^{-j\Omega n_0}X(\Omega)$
Frequency shifting	$e^{j\Omega_0 n}x[n]$	$X(\Omega-\Omega_0)$
Conjugation	$x^*[n]$	$X^*(-\Omega)$
Time reversal	$x[-n]$	$X(-\Omega)$
Time scaling	$x_{(m)}[n]=\begin{cases} x[n/m] & \text{if } n=km \\ 0 & \text{if } n\neq km \end{cases}$	$X(m\Omega)$
Frequency differentiation	$nx[n]$	$j\dfrac{dX(\Omega)}{d\Omega}$
First difference	$x[n]-x[n-1]$	$(1-e^{-j\Omega})X(\Omega)$
Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\pi X(0)\delta(\Omega)+\dfrac{1}{1-e^{-j\Omega}}X(\Omega)$ $\lvert\Omega\rvert\leq\pi$
Convolution	$x_1[n]*x_2[n]$	$X_1(\Omega)X_2(\Omega)$
Multiplication	$x_1[n]x_2[n]$	$\dfrac{1}{2\pi}X_1(\Omega)\otimes X_2(\Omega)$
Real sequence	$x[n]=x_e[n]+x_o[n]$	$X(\Omega)=A(\Omega)+jB(\Omega)$ $X(-\Omega)=X^*(\Omega)$
Even component	$x_e[n]$	$\mathrm{Re}\{X(\Omega)\}=A(\Omega)$
Odd component	$x_o[n]$	$j\,\mathrm{Im}\{X(\Omega)\}=jB(\Omega)$
Parseval's relations		

$$\sum_{n=-\infty}^{\infty} x_1[n]x_2[n]=\frac{1}{2\pi}\int_{2\pi} X_1(\Omega)X_2(-\Omega)\,d\Omega$$

$$\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2=\frac{1}{2\pi}\int_{2\pi} \lvert X(\Omega)\rvert^2\,d\Omega$$

Discrete Fourier Transform

- We often do not have an infinite amount of data which is required by DTFT
  - For example in a computer we cannot calculate uncountable infinite (continuum) of fre TFT

$$x_0[n] = x[n]w_R[n]$$

- Thus, we use DTF to look a a
  - We only observe

$$w_R[n] = \begin{cases} 1, & n = 0, 1, \cdots, N-1 \\ 0, & \text{otherwise} \end{cases}$$

  - In this case the xo[n] is just a sampled data between n=0, n=N-1 (N points)

Discrete Fourier Transform

$$X[k] = X_0(\frac{2\pi k}{N})$$

for $\Omega = \frac{2\pi k}{N}, k = 0, 1, \ldots, N-1$, i.e. only look at the N distinct sampled frequencies of $X_0(\Omega)$.

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, k = 0, 1, \cdots, N-1$$

- Note that in this case the points are spaced 2pi/N, thus the resolution of the samples of the frequency spectrum is 2pi/N.

$$\begin{aligned} X[k] &= X_0(\Omega)|\Omega = \frac{2\pi k}{N}, \ k = 0, 1, \cdots, N-1 \\ &= \sum_{n=0}^{N-1} x[n] e^{-j\Omega n}|_{\Omega = \frac{2\pi k}{N}, k = 0, 1, \cdots, N-1} \\ &= \boxed{\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}}, k = 0, 1, \cdots, N-1} \end{aligned}$$

- We can think series

Let $W_N = e^{-j\frac{2\pi}{N}} \Rightarrow N^{th}$ root of unity $(W_N^N = 1)$ since $W_N^N = e^{-j2\pi} = 1$. You may also write W_N simply as W.

Then

$$\boxed{X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, k = 0, 1, \cdots, N-1}$$

remember:

$$\sum_{n=0}^{N-1} (e^{\frac{-j2\pi k}{N}})^n = \sum_{n=0}^{N-1} W^{kn}$$

Inverse of DFT

$$\boxed{x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn},\ n = 0, 1, \ldots, N-1}$$

- We can obtain the inverse of DFT

$$\sum_{k=0}^{N-1} W_N^{k(l-n)} = \begin{cases} N, & l = n \\ 0, & l \neq n \end{cases}$$

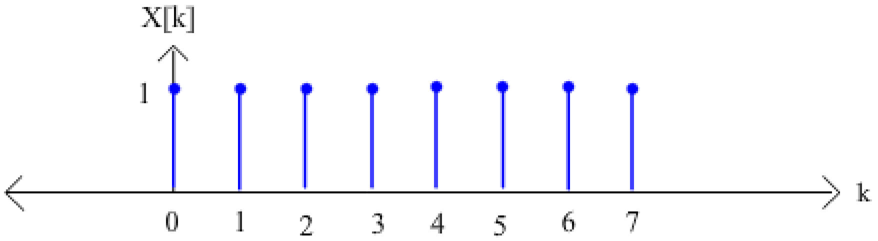
- Note that

Example of DFT

$$x[n] = \begin{cases} 1, & n = 0 \\ 0, & n = 1, \ldots, 7 \end{cases}$$

- Find X[k]
  - We know

$$X[k] = \sum_{n=0}^{7} x[n] W_N^{kn} = \sum_{n=0}^{7} \delta[n] W_N^{kn} = 1 \quad , \forall \, k$$



Example of DFT

Given $y[n] = \delta[n-2]$ and $N = 8$, find $Y[k]$.

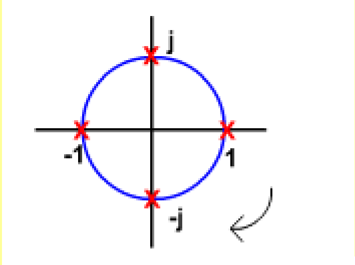
$$Y[k] = \sum_{n=0}^{7} \delta[n-2] W_N^{kn} = W_N^{2k} = e^{\frac{-j2\pi 2k}{N}} = e^{\frac{-j\pi k}{2}}$$

because N = 8

$$= (-j)^k$$

$$Y[k] = [1, -j, -1, j, 1, -j, -1, j]$$

$$x[n-n_0] \longleftrightarrow W_N^{n_0k} X[k]$$



Polar plot for W_N^{2k}

Time shift Property of DFT

$$x[n-n_0]_{\text{mod } N} \leftrightarrow W_N^{kn_0} X[k]$$

Example of DFT

Given $x[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + \delta[n-3]$ and $N = 4$, find $X[k]$.

Summation for X[k]

$$X[k] = \sum_{n=0}^{3} x[n] W_4^{kn} = 1 + 2W_4^{k} + 3W_4^{2k} + W_4^{3k}$$

$$W_4 = e^{\frac{-j\pi}{2}}$$

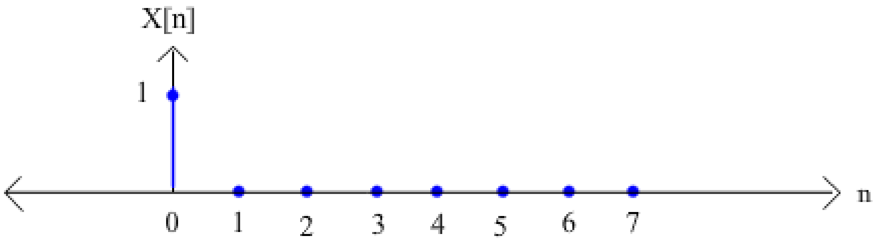
$$X[k] = 1 + 2\,e^{\frac{-j\pi k}{2}} + 3\,e^{-j\pi k} + e^{\frac{-j3\pi k}{2}}$$

Using the shift property!

Example of IDFT

Find the IDFT of $X[k] = 1, k = 0, 1, \ldots, 7$.

$$x[n] = \frac{1}{8}\sum_{n=0}^{7} W_N^{-kn} = \frac{1}{8} N\,\delta[n] = \delta[n]$$

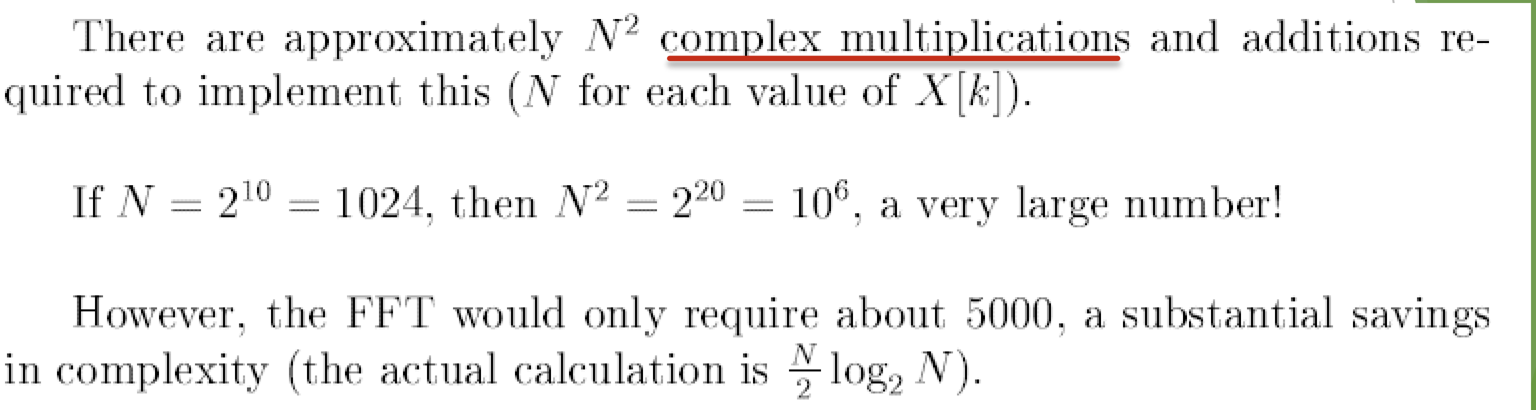


Remember:

$$\boxed{x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] W_N^{-kn}, \; n = 0, 1, \ldots, N-1}$$

Fast Fourier Trans ms

$$X[k] = \sum_{n=0}^{N-1} x[n] W^{kn}$$

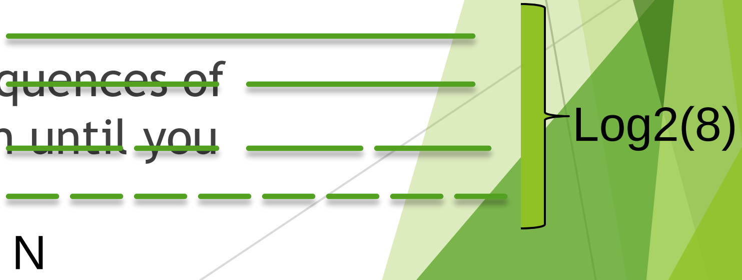


There are approximately N^2 complex multiplications and additions required to implement this (N for each value of $X[k]$).

If $N = 2^{10} = 1024$, then $N^2 = 2^{20} = 10^6$, a very large number!

However, the FFT would only require about 5000, a substantial savings in complexity (the actual calculation is $\frac{N}{2}\log_2 N$).

- Basic idea is to split the sum into 2 subsequences of length N/2 and continue all the way down until you have N/2 subsequences of length 2



Radix-2 FFT Algorithms - Two point FFT

$$\boxed{Y[k] = \sum_{n=0}^{1} y[n] W_2^{kn}} = y[0] + W_2^k y[1]$$

re

$$W_2 = e^{-\frac{j2\pi}{2}} = e^{-j\pi} = -1$$

So we get,

$$Y[k] = y[0] + (-1)^k y[1]$$

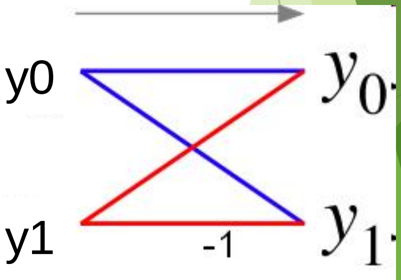
and:

$$Y[0] = y[0] + y[1]$$

$$Y[1] = y[0] - y[1]$$

Advantage: Less computationally intensive: N/2.log(N)

Butterfly FFT



http://www.cmlab.csie.ntu.edu.tw/cml/dsp/training/coding/transform/fft.html

General FFT Algorithm

- First break x[n] into even and odd

$$X[k] = \sum_{n=0}^{N-1} x[n]W^{kn}$$

$$X[k] = \sum_{n\,\mathrm{even}} x[n]W^{kn} + \sum_{n\,\mathrm{odd}} x[n]W^{kn}$$

- Let n=2m for even and n=2m+1 for odd
- Even and odd parts are both DFT of a N/2 point sequence

$$X[k] = \sum_{m=0}^{\frac{N}{2}-1} x[2m]W^{2mk} + \sum_{m=0}^{\frac{N}{2}-1} x[2m+1]W^{k(2m+1)} =$$

$$\boxed{\sum_{m=0}^{N/2-1} W_{N/2}{}^{mk} x[2m] + W_N{}^{k}\left(\sum_{m=0}^{N/2-1} W_{N/2}{}^{mk} x[2m+1]\right)}$$

$$W_N{}^{2mk} = W_{N/2}{}^{mk}$$

$$W_{N/2}{}^{m+N/2} = W_{N/2}{}^{m} W_{N/2}{}^{N/2} = W_{N/2}{}^{m}$$

$$W_N{}^{N} = e^{-2\pi j} = \cos(-2\pi) - j\sin(-2\pi) = 1$$

$$W_N{}^{N/2} = -1$$

- Break up the size N/2 subsequent in half by letting 2m→m
- The first subsequence here is the term x[0], x[4], …
- The second subsequent is x[2], x[6], …

Example

Let's take a simple example where only two points are given n=0, n=1; N=2

$$X[k]=\sum_{m=0}^{N/2-1} W_{N/2}^{mk}\, x[2m] + W_N^{k}\Big(\sum_{m=0}^{N/2-1} W_{N/2}^{mk}\, x[2m+1]\Big)$$

$$W_N^{2mk}=W_{N/2}^{mk}$$

$$W_{N/2}^{m+N/2}=W_{N/2}^{m}W_{N/2}^{N/2}=W_{N/2}^{m}$$

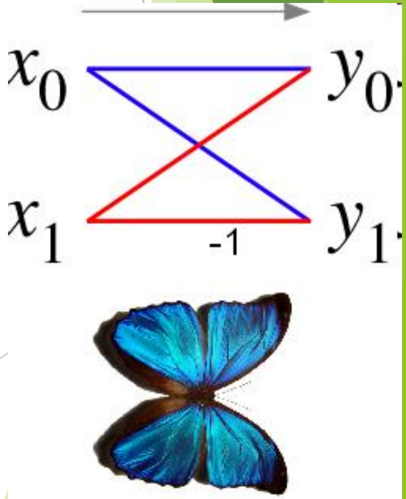
$$W_N^{N}=e^{-2\pi j}=\cos(-2\pi)-j\sin(-2\pi)=1$$

$$W_N^{N/2}=-1$$

$$X[k=0]=\sum_{m=0}^{0} W_1^{0.0}x[0]+W_1^{0}\Big(\sum_{m=0}^{0} W_1^{0.0}x[1]\Big)=x[0]+x[1]$$

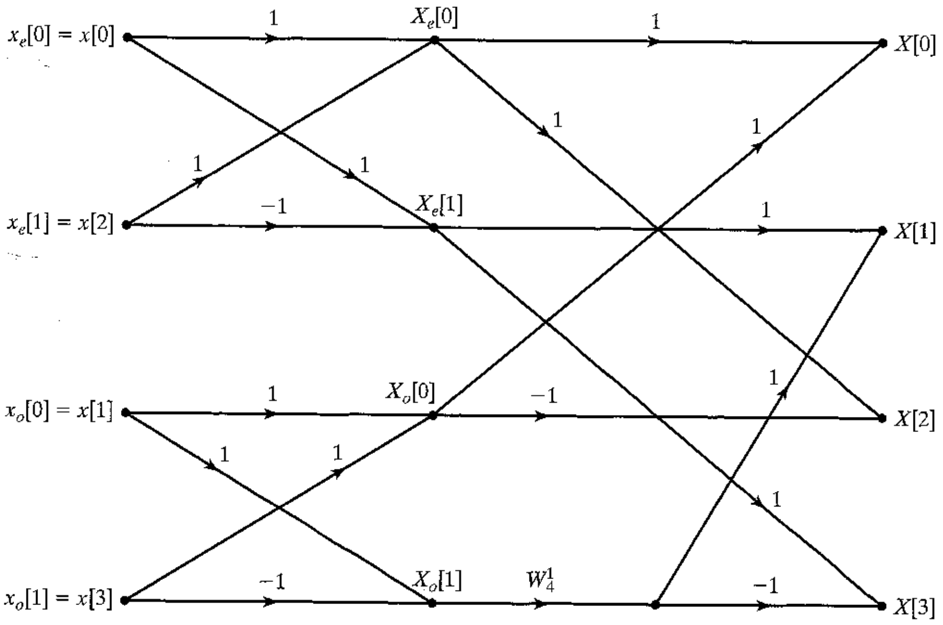
$$X[k=1]=\sum_{m=0}^{0} W_1^{0.1}x[0]+W_1^{1}\Big(\sum_{m=0}^{0} W_1^{0.1}x[1]\Big)=x[0]+W_1^{1}x[1]=x[0]-x[1]$$

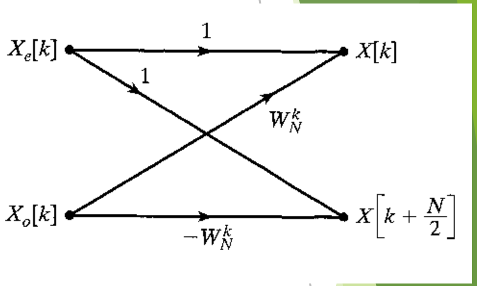
Same result



FFT Algorithms - Four point FFT

First find even and odd parts and then combine them:

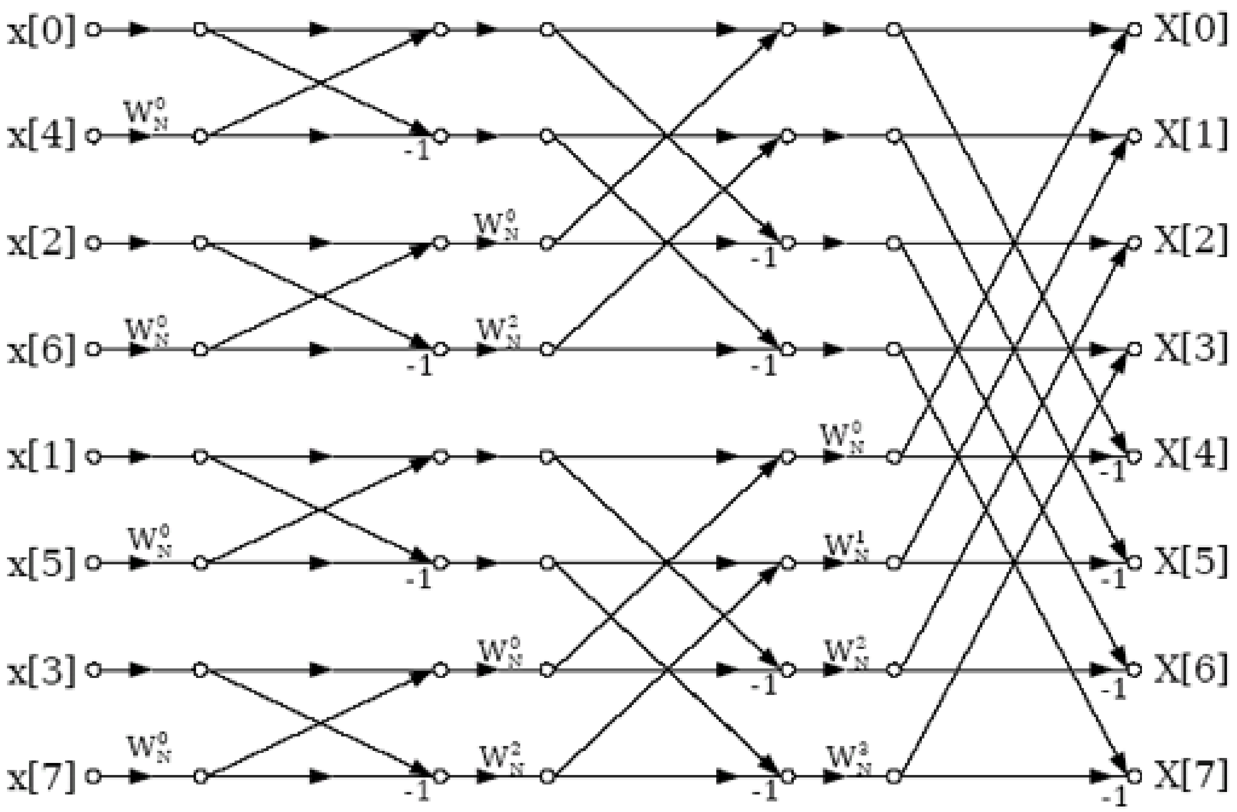




The general form:

$$\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \end{bmatrix} = \begin{bmatrix} +1 & +1 & +1 & +1 \\ +1 & -j & -1 & +j \\ +1 & -1 & +1 & -1 \\ +1 & +j & -1 & -j \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{bmatrix}$$

FFT Algorithms - 8 point FFT


x[0]
x[4]
x[2]
x[6]
x[1]
x[5]
x[3]
x[7]
W$_N^0$
W$_N^0$
W$_N^0$
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X[0]
X[1]
X[2]
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X[4]
X[5]
X[6]
X[7]